J. Appl. Maths Mechs Vol. 55, No. 6, pp. 910–920, 1991
Printed in Great Britain.

0021–8928/91 $15.00+.00

VARIATIONAL FORMULATIONS FOR DISCONTINUOUS DISPLACEMENT FIELDS IN PROBLEMS OF THE DEFORMATION THEORY OF PLASTICITY WITHOUT HARDENING†

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(*Received* 23 *October* 1990)

The problem of the construction and use of extended variational formulations which enable an explicit analysis to be made of discontinuous displacement fields for a wide class of problems of the deformation theory of plasticity is discussed. Three-dimensional, as well as plane problems with the Mises and Schleicher–Moreau criteria are investigated. In the case of a piecewise-continuous discontinuity line it is shown that the existence of a saddle point of an extended Lagrangian results in an integral inequality, which imposes certain conditions on the trace of the stress tensor on the line of discontinuity. Different arguments were used in [1–3] to obtain different versions of this condition for a number of problems of the theory of plasticity. When sufficient regularity of the stresses is assumed, then from the condition in question a simple algebraic relation follows connecting, at the line of discontinuity, the value of the stress tensor with the parameters determining the magnitude and direction of the discontinuity. Examples are given, which show that, generally speaking, only some of the stress states lying on the yield surface correspond to discontinuous solutions.

In a number of papers (see in particular [4–11]) the variational formulation of the problems of deformation theory of ideal plasticity have been considered. It is known that these formulations have a number of special features. Thus, the problem of stresses consists here of minimizing a quadratic functional on a set of statically admissible stress fields satisfying the yield conditions, and when the set is non-empty the problem always has a unique solution [4]. The dual of this problem will be the problem of minimizing a convex functional on the set of admissible displacement fields. Such a variational formulation is found to be suitable, unlike the previous extremal problem without constraints, for numerical analysis in the case when the existence of its solution is guaranteed. Appropriate examples, however, show that the proposed formulation is mathematically incorrect since a discontinuous solution may occur on which the starting functional is not defined [5]. Thus the need arises to construct the extended (relaxed) problem which retains the value of the exact lower limit, thus making it possible to take into account all limiting elements of the initial formulation.

Complete variational extensions for the problems of deformation theory with the Mises yield criterion were constructed ([5–9]; see also the bibliography quoted in these papers). The mathematical formulation, however, of the total variational extensions are very abstract. For

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 1026–1034, 1991.

example, the space of functions of limited deformation on which the extended functional is defined in [7–9], consists of summable vector functions for which the strain tensor is the Radon measure. Therefore, from the practical point of view it is more convenient to use so-called partial extensions, in which the functional is defined on the functions which have first order discontinuities along certain surfaces (lines). Partial extensions are simple and can be used efficiently in numerical solutions of the problems [12–14].

The main portion of the present paper deals with constructing the partial extensions for a wide class of problems of the deformation theory of plasticity.

1. The problem of determining the stresses σ and displacements u can be reduced, within the framework of deformation theory, to solving the system

$$\begin{gathered}\sigma_{ij,j}+f_i=0 \text{ in } \Omega\\ u_i=u_i^0 \text{ on } \Gamma_1,\ \sigma_{ij}n_j=F_i \text{ on } \Gamma_2\\ \varepsilon_{ij}(u)=A_{ijkl}\sigma_{kl}+\lambda_{ij},\ 2\varepsilon_{ij}=u_{i,j}+u_{j,i}\end{gathered} \tag{1.1}$$

under the following conditions:

$$\lambda_{ij}(\tau_{ij}-\sigma_{ij})\leqslant 0,\ \forall\tau\in M_c^k,\ G(\tau)\leqslant 0 \tag{1.2}$$

Here $\Omega\in R^k$ $(k=2, 3)$ is the region with the boundary Γ which is Lipshitz-continuous, occupied by an elastoplastic body, A_{ijkl} are the components of the elasticity tensor, f, F are the volume and surface force vectors, n is the outer normal to f, F is the set of symmetric tensors of dimension k, $G: M_c^k\to R^1$ is a convex function governing the plastic properties of the material, and the convention of summation over repeated indices is observed. We will assumed that

$$\begin{gathered}\Gamma=\Gamma_1\cup\Gamma_2,\ \Gamma_1\cap\Gamma_2=\varnothing\\ f\in(L^2(\Omega))^k,\ F\in(L^2(\Gamma_2))^k,\ u^0\in(H^1(\Omega))^k\end{gathered} \tag{1.3}$$

Let us introduce into our discussion the Hilbert space

$$\Sigma=\{\sigma\in M_c^k: \sigma=\{\sigma_{ij}\},\ \sigma_{ij}\in L^2(\Omega)\}$$

where the scalar product is defined as $(\sigma,\tau)=\int_\Omega\sigma_{ij}\tau_{ij}dx$.

We introduce the sets

$$\begin{gathered}\Sigma_0=\{\sigma\in\Sigma: \sigma_{ij,j}\in L^2(\Omega)\}\\ U=\{u: u\in(H^1(\Omega))^k,\ u=u^0 \text{ on } \Gamma_1\}\\ M=\{\sigma\in\Sigma_0: \sigma_{ij,j}+f_i=0 \text{ in } \Omega,\ n_i\sigma_{ij}=F_j \text{ on } \Gamma_2\}\end{gathered}$$

The set M contains statically admissible stress fields. Let us find the set of tensors satisfying the yield condition

$$K=\{\sigma\in\Sigma: G(\sigma)\leqslant 0 \text{ a.e. in } \Omega\}$$

Then, if the set $K\cap M$ is non-empty and u^*, σ^* is a solution of problem (1.1)–(1.2), the stress field σ^* will minimize on $K\cap M$ the functional

$$\begin{gathered}\Phi(\sigma)=a(\sigma,\sigma)-\int_{\Gamma_1}n_i\sigma_{ij}u_j^0d\Gamma\\ a(\sigma,\sigma)=\frac{1}{2}\int_\Omega A_{ijkl}\sigma_{ij}\sigma_{kl}\,dx\end{gathered} \tag{1.4}$$

It was proved (see e.g. [4]) that if $K\cap M\neq\varnothing$, then a solution of this problem exists and is unique. However, the use of (1.4) to solve specific problems meets with difficulties, since the minimization is carried out not over the whole space, but on the set $K\cap M$, i.e. on the set of tensors satisfying the constraints in the form of equations (the equations of equilibrium and the boundary load conditions) and inequalities (the yield conditions). Moreover, a problem arises in constructing the displacement field corresponding to the stress field. We note that within the framework of the initial formulation such a displacement field need not exist.

Problem (1.1), (1.2) can also be formulated as the problem of determining the saddle point of the Lagrangian

$$l(\sigma, u) = \int_\Omega (\varepsilon_{ij}(u)\sigma_{ij} - f_i u_i)\,dx - a(\sigma,\sigma) - \int_{\Gamma_2} F_i u_i\,d\Gamma$$

It can be shown that the problem of minimizing the functional (1.4) on $K\cap M$ is equivalent to the problem

$$\sup_{\sigma\in K}\inf_{u\in U} l(\sigma, u)$$

and (see [4])

$$\sup_{\sigma\in K}\inf_{u\in U} l(\sigma, u) = \inf_{u\in U}\sup_{\sigma\in K} l(\sigma, u) = C \tag{1.5}$$

so that the first component of the saddle point exists and is identical with σ^*. Calculating the supremum in σ on the right-hand side of Eq. (1.5), we arrive at the dual problem which will represent the problem of minimizing the convex functional

$$J(u) = \sup_{\sigma\in K} l(\sigma, u)$$

on the set U [15]. For example, if the material is isotropic and $G(\sigma)=|\sigma^D|^2-2k_*^2$ (the Mises criterion) where σ^D is the deviator of the tensor σ, $|\sigma|=(\sigma_{ij}\sigma_{ij})^{1/2}$, k_* is the yield point, then

$$J(u) = \int_\Omega \left[\frac{1}{2}k_0(\operatorname{div} u)^2 + H(|\varepsilon^D(u)|) - f_i u_i\right]dx - \int_{\Gamma_2} F_i u_i\,d\Gamma \tag{1.6}$$

$$H(t) = \begin{cases}\mu t^2, & t\leqslant k_*/(\sqrt{2}\mu)\\ k_*(\sqrt{2}t - k_*/(2\mu)), & t > k_*/(\sqrt{2}\mu)\end{cases}$$

Here ε^D is the strain tensor deviator and k_0, μ are the elastic constants of the material.

If the problem

$$\inf_{u\in U} J(u) \tag{1.7}$$

has a solution u^*, then the pair (u^*, σ^*) will be a saddle point $l(\sigma, u)$, and conversely, if $l(\sigma, u)$ has a saddle point its components will be solutions of problems (1.4) and (1.7).

The variational formulation (1.7) can be used when constructing the solution of the problem (1.1), (1.2). It is more suitable, since the minimization is carried out over the whole space U, which is particularly important when variational-difference methods are used. Problem (1.7) however, has a serious drawback: its solution may not exist.

This is due to the fact that $J(u)$ is non-coercive on U and coercive only in a non-reflexive space (the corresponding spaces for the functional (1.6) are derived in [5]). These difficulties arose due to the possibility of discontinuous solutions occurring on which $J(u)$ is not defined. Therefore, the need arises to construct a variational extension of the given class of problems which would bring into our discussion all limit functions of the initial set. These problems have been studied recently in great detail (see [5–8], where the corresponding abstract extensions were given and an extended formulation was obtained in [5] for problems with the Mises yield criterion, in which displacement discontinuities are allowed on certain surfaces). It should be noted that complete variational extensions have a sufficiently abstract form and their indirect use in solving the problems meets with difficulties. At the same time, if in extending the set U we restrict ourselves to functions which can have discontinuities only along certain surfaces (or curves in the plane case), we can construct explicitly the corresponding extended variational formulations and use them to solve specific problems and to construct numerical methods.

2. We shall consider the problem of determining the saddle point of the Lagrangian $l(\sigma, u)$: $K\times U\to R^1$, assuming that condition (1.5) holds and the first component of the saddle point σ^* exists, i.e.

$$\sup_{\sigma\in K} \Phi(\sigma) = \Phi(\sigma^*);\quad \Phi(\sigma) = \inf_{u\in U} l(\sigma, u) \tag{2.1}$$

We shall construct the Lagrangian $l'(\sigma, u)$: $K\times U'\to R^1$ which must be identical with $l(\sigma, u)$ on $K\times U$, retain the formulation (2.1) as the problem for the variable σ, and have the property that, if (σ^*, u^*) is a saddle point of $l'(\sigma, u)$, then it can be approximated by the sequence of elements of $K\times U$. To do this, we shall consider the Banach space V, such that the set U is imbedded continuously and densely everywhere in V, and $U\subset U'\subset V$.

We shall require that the following conditions hold:

$$\begin{aligned}
&1^\circ.\ l'(\sigma, u) = l(\sigma, u),\ \forall u\in U,\ \forall\sigma\in K\\
&2^\circ.\ \inf_{u\in U'} l'(\sigma, u) = \inf_{u\in U} l(\sigma, u),\ \forall\sigma\in K\\
&3^\circ.\ \forall u'\in U'\ \exists\ \{u_m\}\in U:\ u_m\to u' \text{ в } V_w\\
&\qquad \lim_{m\to\infty} l(\sigma, u_m) \leqslant l'(\sigma, u')\ \forall\sigma\in K\cap M
\end{aligned} \tag{2.2}$$

Conditions (2.2) show that $l'(\sigma, u)$ is a continuation of $l(\sigma, u)$ onto $K\times U'$, with the variational problem (2.1) retained.

Assertion 1. When conditions (2.2) hold, the Lagrangian $l'(\sigma, u)$ has the following properties.

Case 1. If (σ^*, u^*) is a saddle point of $l(\sigma, u)$ on $K\times U$, then (σ^*, u^*) is a saddle point of $l'(\sigma, u)$ on $K\times U'$

Case 2. $\sup_{\sigma\in K} \inf_{u\in U'} l'(\sigma, u) = \inf_{u\in U'} \sup_{\sigma\in K} l'(\sigma, u) = C$

Case 3. If (σ^*, u^*) is a saddle point of $l'(\sigma, u)$ on $K\times U'$, then a sequence $\{u_m\}\in U$ exists such that $u_m\to u^*$ in V and

$$\lim_{m\to\infty} l(\sigma^*, u_m) = l'(\sigma^*, u^*) \tag{2.3}$$

Proof. From (2.2) it follows that

$$\sup_{\sigma\in K}\inf_{u\in U'} l'(\sigma, u) = \sup_{\sigma\in K}\inf_{u\in U} l(\sigma, u) = C$$
$$\inf_{u\in U'}\sup_{\sigma\in K} l'(\sigma, u) \leqslant \inf_{u\in U}\sup_{\sigma\in K} l(\sigma, u) = C$$

and this yields the first two points of the assertion. Let us assume that (σ^*, u^*) is a saddle point of $l'(\sigma, u)$. Then from the last condition of (2.2) it follows that there exists a sequence $\{U_m\}\in U$ such that $u_m \to u^*$ in V and

$$\lim_{m\to\infty} l(\sigma^*, u_m) \leqslant l'(\sigma^*, u^*) = C$$

On the other hand

$$l(\sigma^*, u_m) = l'(\sigma^*, u_m) \geqslant \inf_{u\in U'} l'(\sigma^*, u) = C$$

Therefore we have relation (2.5).

Using the Lagrangian $l'(\sigma, u)$, we can construct the problem

$$\inf_{u\in U'} J'(u);\ J'(u) = \sup_{\sigma\in K} l'(\sigma, u) \tag{2.4}$$

and the problem (2.1) will also be dual to it. From the known properties of saddle points it follows that if problem (2.4) has a solution u^*, then (σ^*, u^*) will be a saddle point of $l'(\sigma, u)$ on $K\times U'$, and conversely, if (σ^*, u^*) is a saddle point of $l'(\sigma, u)$, then u^* will be a solution of problem (2.4).

By virtue of the first condition of (2.2)

$$J'(u) = J(u)\ \forall u\in U$$
$$\inf_{u\in U'} J'(u) = \inf_{u\in U'}\sup_{\sigma\in K} l'(\sigma, u) = C = \inf_{u\in U} J(u) \tag{2.5}$$

The above inequalities show that the functional $J'(u)$ constructed in this manner is a continuation of $J(u)$ onto a wider set U', with the exact lower limit of the initial problem retained. It also follows from relations (2.5) that any minimizing sequence in problem (1.7) will also be minimizing for (2.4), and if the problem (1.7) has a solution, this solution will also be a solution of problem (2.4).

Assertion 1 shows that the problems for the Lagrangians $l(\sigma, u)$ and $l'(\sigma, u)$ are closely connected. The first component of the saddle point exists in both cases and is defined uniquely. The second component must satify the conditions

$$l(\sigma, u^*) \leqslant l(\sigma^*, u^*) \leqslant l(\sigma^*, u),\ \forall u\in U,\ \forall\sigma\in K \tag{2.6}$$
$$l'(\sigma, u^*) \leqslant l'(\sigma^*, u^*) \leqslant l'(\sigma^*, u),\quad \forall u\in U',\quad \forall\sigma\in K$$

If $u^*\in U'$ and $u^*\notin U$, then $l'(\sigma, u)$ has a saddle point and the second condition of (2.6) holds. At the same time the first condition of (2.6) does not hold, since the Lagrangian $l(\sigma, u)$ is not defined on u^*. In this sense, there is no displacement field from U corresponding to the stress field σ^* which is a solution of the problem, but there is a displacement field from the wide class U'. The latter can be approached as closely and accurately as required by the fields $\{u_m\}$, which are

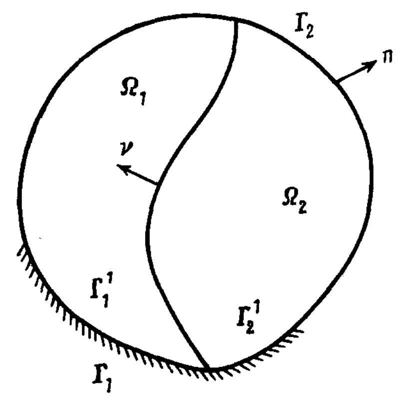

FIG. 1.

admissible in the initial formulation. This, in fact, means that the extended formulation is simply a more correct formulation of the problem, in which the special features, which are present implicitly in the initial formulation, now appear in explicit form.

3. Let us consider the simplest case, when $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \varnothing$ (see Fig.1) where Ω_1, Ω_2 are open regions whose boundaries $\partial\Omega_1$ and $\partial\Omega_2$ are Lipshitz continuous and $\gamma = \partial\Omega_1 \cap \partial\Omega_2$. Let us write $\Gamma_i^s = \Gamma_i \cap \partial\Omega_s$, $(i, s = 1, 2)$ and introduce the sets

$$U' = \{u(x): u(x) = u^s(x) \text{ for } x \in \Omega_s,\ u^s(x) \in U,\ s = 1, 2\}$$

$$K_0 = \{\tau \in \Sigma_0: G(\tau) \leqslant 0 \text{ a.e. in } \Omega\}$$

Let $V = L^p(\Omega)$, $p > 1$, ν is the vector of the normal to the surface (line) γ, $v = u^1 - u^2$, $\tau \in K_0$. Let us write

$$[\tau, v] = \int_\gamma \nu_i \tau_{ij} v_j \, d\Gamma; \quad R_\gamma(v) = \sup_{\tau \in K_0} [\tau, v] \tag{3.1}$$

We note that since the trace v on the line γ belongs to the space $H^{1/2}$ and $\tau \in K_0$, it follows that the above expression is meaningful. Let us define the extended Lagrangian as follows:

$$l'(\sigma, u) = L_\gamma(\sigma, u) + R_\gamma(v)$$

$$L_\gamma(\sigma, u) = \sum_{s=1}^{2} \left[\int_{\Omega_s} (e_{ij}(u^s)\sigma_{ij} - u_i^s f_i) \, dx - \int_{\Gamma_2^s} u_i^s F_i \, d\Gamma \right] - a(\sigma, \sigma)$$

We see that $U \subset U' \subset V$, and if $u^1 = u^2 = u(x)$, $u(x) \in U$, then $l'(\sigma, u) = l(\sigma, u)$ so that the first condition of (2.2) holds. To verify the second condition we shall use the inequality

$$\inf_{u \in U'} l'(\sigma, u) \leqslant \inf_{u \in U} l(\sigma, u) = \inf \left[\int_\Omega (e_{ij}(u)\sigma_{ij} - u_i f_i) \, dx - \int_{\Gamma_2} u_i F_i \, d\Gamma \right] - a(\sigma, \sigma)$$

The expression within the square brackets represents a linear functional in u, and its infimum is different from $-\infty$ only when $\sigma \in M$. If on the other hand $\sigma \in M$, then integrating by parts we obtain

$$L_\gamma(\sigma, u) = \Phi(\sigma) - [\sigma, v]$$

Since

$$\inf_{u\in U'}\sup_{\tau\in K_*}[\tau-\sigma, v] \leqslant \inf_{u\in U}\sup_{\tau\in K_*}[\tau-\sigma, v] = 0$$
$$\sup_{\tau\in K_*}\inf_{u\in U'}[\tau-\sigma, v] \geqslant 0$$

then

$$\inf_{u\in U'} l'(\sigma, u) = \begin{cases}\Phi(\sigma) & \text{if } \sigma\in M\\ -\infty & \text{otherwise}\end{cases}$$

To verify the last condition of (2.2) we construct the sequence $v_m(x) = u^2 + \varphi_m(x)(u^1 - u^2)$, where $\varphi_m(x)$ is a smooth function and $0\leqslant\varphi_m(x)\leqslant 1$, $\varphi_m(x) = 1$ in Ω_1, $\varphi_m(x) = 0$ and Ω_{2m} where

$$\Omega_{2m} = \{x\in\Omega_2\colon \text{dist}\,(x, \partial\Omega_2) > h/m,\ h = \text{const}\}$$

When $m\to\infty$, we have $\varphi_m(x)\to\chi(\Omega_2)$ where χ is the characteristic function of the set. Then

$$v_m\in U,\ v_m(x)\to u'(x)bV,\ u'(x)\in U',\ u'(x) = u^s(x)$$

for $x\in\Omega_s$ ($s = 1, 2$) and

$$l(\sigma, v_m) = \sum_{s=1}^{2}\Big[\int_{\Omega_s}\varepsilon_{ij}(u^s)\sigma_{ij}\,dx - \int_{\Gamma_2^s} u_i^s F_i\,d\Gamma\Big] -$$
$$-\int_{\Omega}(v_m)_i f_i\,dx - a(\sigma,\sigma) + \int_{\omega_m}\varepsilon_{ij}(v_m - u^2)\sigma_{ij}\,dx,$$

where $\omega_m = \Omega\backslash\Omega_{2m}$. Integrating by parts, we obtain the following expression for the last term:

$$[\sigma, v] - \int_{\omega_m}\varphi_m v_i \sigma_{ij,j}\,dx$$

The first term is independent of m and the second term tends, by virtue of absolute continuity of the Lebesque integral, to zero as $m\to\infty$. Therefore

$$\lim_{m\to\infty} l(\sigma, v_m) = L_\gamma(\sigma, u') + [\sigma, v] \leqslant l'(\sigma, u')$$

Thus conditions (2.2) hold and hence Assertion 1 holds for $l'(\sigma, u)$.

Using the Lagrangian $l'(\sigma, u)$ we can now construct the extended problem

$$\inf_{u\in U'} J'(u'),\ J'(u) = J_\gamma(u) + R_\gamma(v) \tag{3.2}$$
$$J_\gamma(u) = \sup_{\sigma\in K} L_\gamma(\sigma, u),\ v = u^1 - u^2$$

The term $R_\gamma(v)$ represents the "penalty" for the break in the displacement field on the line γ, and is calculated from (3.1). Here the supremum can be conveniently calculated on the set of tensors τ continuous in the neighbourhood of γ, which form in K_0 a set which is dense everywhere.

If we define in the tensor space M_c^k the function

$$\Psi(e) = \sup_{G(\tau)\leqslant 0} \tau_{ij}e_{ij}, \quad \Psi: M_c^k \to R^1 \tag{3.3}$$

which is the support function of the convex set $\{\tau: \tau \in M_c^k, G(\tau) \leqslant 0\}$, then

$$R_\gamma(v) = \int_\gamma \Psi(e(v))\,dx, \quad 2e_{ij} = \nu_i v_j + \nu_j v_i \tag{3.4}$$

By virtue of the properties of support functions, this implies that $R_\gamma(v)$ is a positively homogeneous and subadditive function of v.

Assertion 2. If there exists a saddle point $(\sigma^*, u^*) \in K \times U'$ of the Lagrangian $l'(\sigma, u)$ and $u^* \subset u^s(x)$ when $x \in \Omega_s$, $u^s(x) \in U$ $(s = 1, 2)$, then the trace of tensor σ^* on the line of discontinuity of displacements γ must satisfy the condition

$$[\tau - \sigma^*, v] = \int_\gamma (\tau_{ij} - \sigma_{ij}^*)\, e_{ij}(v)\, d\Gamma \leqslant 0, \quad \forall \tau \in K_0$$
$$(v = u^1 - u^2) \tag{3.5}$$

Proof. If (σ^*, u^*) is a saddle point, then

$$L_\gamma(\sigma^*, u^*) + \sup_{\tau \in K_0} [\tau, v^*] \leqslant L_\gamma(\sigma^*, w) + \sup_{\tau \in K_0} [\tau, w'] \ \forall w \in U'$$
$$w(x) = w^s(x), \text{ if } x \in \Omega_s, \ w^s(x) \in U, \ s = 1, 2, \ w' = w^1 - w^2$$

Integrating by parts and taking into account the fact that $\sigma^* \in K \cap M$, we have

$$\sup_{\tau \in K_0} [\tau - \sigma^*, v] \leqslant \sup_{\tau \in K_0} [\tau - \sigma^*, w'], \ \forall w \in U'$$

Taking the upper limit of the right-hand side of this inequality with respect to the function $w \in U$, we obtain (3.5).

From (3.5) it follows that if $G(\sigma^*) < 0$, then $e(v) \equiv 0$, i.e. no discontinuities are possible in the elastic region. If $G(\sigma^*) = 0$ and the trace of the tensor σ^* is a function defined at almost every point of γ (this will occur, for example, when $\sigma_{ij}^* \in W_p^1(\Omega)$, $p > 1$), then at almost every point of γ the tensor $e(v)$ will have to be directed along the normal to the yield surface $G(\sigma) = 0$ at the point $\sigma = \sigma^*$ (for a non-smooth surface $e(v)$ must belong to the corresponding cone of subnormals). Also, if G is a differentiable function, then (3.5) will lead to the well-known relation [1]

$$e_{ij}(v) = \lambda g_{ij}(\sigma^*), \quad g_{ij} = \partial G / \partial \sigma_{ij}, \quad \lambda \geqslant 0 \tag{3.6}$$

The existence of a discontinuity $v(x) \neq 0$, $x \in \gamma$ is possible only in the case when the following system has a solution at the point x:

$$\nu_i v_j + \nu_j v_i = g_{ij}(\sigma^*), \quad |\nu| = 1, \quad G(\sigma^*) = 0 \tag{3.7}$$

It turns out that in the general case system (3.7) does not have a solution for all σ^* lying on the yield surface. For example, in the problem of the plane stress state with the Mises yield criterion there is no solution when $0.5 < \sigma_1/\sigma_2 < 2$ (σ_1, σ_2 are the principal stresses), which corresponds to the region of ellipticity [16] of the system of equations in the stresses. In the problem of plane deformation with the Mises criterion system (3.7) always has a solution, and its analysis leads to a well-known argument about the possibility of discontinuities occurring along the characteristics.

If we choose, in the three-dimensional problem with a yield criterion, the axes of a Cartesian system of coordinates which coincide with the directions of the principal axes of the tensor σ^* at the point $x \in \gamma$, then from system (3.7) it follows that only a tangential discontinuity is possible. Discontinuous solutions will be possible only when the deviator σ^* in the given system of coordinates is a diagonal tensor all of whose diagonal components are different and take one of the following three values: 0, k_*, $-k_*$. Six straight lines lying on the surface of the Mises cylinder and parallel to its axis correspond to these stress states.

Analysis of condition (3.7) in the axisymmetric problem with Mises criterion shows that the line of discontinuity may approach the free surface of a cylindrical body only at an angle of ±45° to the generatrix, which agrees with experimental data [17].

Note. In the case when $\Omega_2 = \Omega$, the extended Lagrangian will have the form

$$l'(\sigma, u) = (\varepsilon(u), \sigma) - a(\sigma, \sigma) - \int_\Omega f_i u_i dx - \int_{\Gamma_2} F_i u_i d\Gamma + R_\gamma(u - u^0)$$

$$R_\gamma(u - u^0) = \sup_{\tau \in K_*} \int_{\Gamma_1} n_i \tau_{ij}(u_j^0 - u_j)\, d\Gamma$$

In this case the term R_γ can be regarded as the penalty for possible violation of the boundary condition on Γ_i. The results of Sec. 3 can be generalized in a natural way to the case when

$$\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_N$$

$$\Omega_s \cap \Omega_t = \varnothing,\ \Gamma_{st} = \partial\Omega_s \cap \partial\Omega_t,\ \Gamma_s^1 = \partial\Omega_s \cap \Gamma_1$$

where $x \in \Omega_s$ $u(x) = u^s(x)$, $u^s(x) \in (H^1(\Omega))^k$, $s, t = 1, 2, \ldots N$.

In this case the sum in the expression for $L_\gamma(\sigma, u)$ is calculated from 1 to N, and R_γ is given by the expression

$$\sup_{\tau \in K_*} \left(\sum_{s,t=1}^{M} \int_{\Gamma_{st}} \nu_i \tau_{ij}(u_j^s - u_j^t)\, d\Gamma + \sum_{s=1}^{N} \int_{\Gamma_s^1} \nu_i \tau_{ij}(u_j^0 - u_j)\, d\Gamma \right)$$

where the corresponding integral is assumed to be equal to zero if Γ_{st} and Γ_s^1 is an empty set.

4. We shall discuss the extended variational formulation which follows from formulas (3.2)–(3.4) for some important cases. We will denote by e_0 and e^D the spherical and deviator part of the tensor e. We note that $e_0 = \nu_i e_i = u_n^1 - u_n^2$ corresponds to the normal component of the vector ν. The extended functional, defined on functions which may undergo a discontinuity along the line γ, can be written in the form

$$J'(u) = \sum_{s=1}^{2} \left\{ \int_{\Omega_s} (H(\varepsilon(u^s)) - f_i u_i^s)\, dx - \int_{\Gamma_2^s} F_i u_i^s\, d\Gamma \right\} + \int_\gamma \Psi(e(v))\, d\Gamma \qquad (4.1)$$

where H, Ψ are governed by the choice of the yield condition.

For a three-dimensional problem with Mises condition $H(\varepsilon)$ is determined in accordance with (1.6) and $\Psi(e) = \sqrt{2} k_* |e^D(v)|$. The argument showing the need to include such a term in the energy relationships is given in [16], and a strict justification for this fact is given in [5, 6] from the positions of variational calculus. (See [12–14] for description of the use of the corresponding extended formulation in constructing the variational-difference methods.)

Let us consider a problem of the plane stress state with Mises yield condition, which can be written, for $\sigma \in M_c^2$, in the form

$$a^2 |\sigma^D|^2 + b^2 \sigma_0^2 \leqslant k_*^2,\ a = 1/\sqrt{2},\ b = 1/\sqrt{12},\ \sigma_0 = \sigma_{11} + \sigma_{22}$$

Then the extended formulation (4.1) will have the simplest form for an incompressible medium ($k_0 = \infty$). In this case

$$H(\varepsilon) = \begin{cases} {}^1/_6 Et^2(\varepsilon), & \text{if } t \leqslant 3k_*/E \\ k_*(t(\varepsilon) - {}^3/_2 k_* E), & \text{if } t > 3k_*/E \end{cases}$$

where $t(\varepsilon) = (2|\varepsilon^D|^2 + 3\varepsilon_0^2)^{1/2}$ and E is Young's modulus. When $k_0 \leqslant \infty$, the function $\Psi(e)$ has the form

$$\Psi(e) = k^*(2|e^D|^2 + 3e_0^2)^{1/2}$$

We know that in many problems (such as soil behaviour analysis, the study of porous media, etc.), yield criteria which take into account the dependence on the first invariant of the stress tensor are used. They are called the Schleicher–Moreau conditions [17, 18]

$$G(\sigma) = |\sigma^D| + h(\sigma_0) \leqslant 0, \quad \sigma \in M_c^3 \tag{4.2}$$

Here h is a convex function determining the dependence of $G(\sigma)$ on σ_0. The Coulomb–Moreau criterion [18] is often used; this is a special case of (4.2) when $h(\sigma_0) = a\sigma_0 - b$ where a, b are constants.

In this case we have the following expressions for H and Ψ:

$$\Psi(e) = \begin{cases} be_0/(3a), & \text{if} \quad e_0 \geqslant 0 \text{ и } |e^D| \leqslant e_0/(3a) \\ +\infty & \text{otherwise} \end{cases}$$

$$H(\varepsilon) = C_1(2\varepsilon_0 - C_2) + H_1(\varepsilon_0', |\varepsilon^D|), \quad \varepsilon_0' = \varepsilon_0 - C_2$$

$$C_1 = b/(6a), \quad C_2 = b/(3ak_0), \quad C_3 = [a^2/\mu + 2/(9k_0)]^{-1}$$

and $H_1 \equiv 0$ if $\varepsilon'_0 > 3a|\varepsilon^D$, otherwise it is given by the following expressions:

$$H_1(\varepsilon) = \begin{cases} \mu|\varepsilon^D|^2 + k_0(\varepsilon_0')^2/2, & \text{if } \varepsilon_0' \leqslant -2\mu|\varepsilon^D|/(3ak_0) \\ C_3(3a|\varepsilon^D| - \varepsilon_0')^2, & \text{if } \varepsilon_0' > -2\mu|\varepsilon^D|/(3ak_0) \end{cases}$$

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Translated by L.K.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 920–926, 1991
Printed in Great Britain.
0021-8928/91 $15.00+.00

THE AXISYMMETRIC STATIC PROBLEM OF THERMOELASTICITY FOR A MULTILAYERED CYLINDER†

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(*Received* 24 *September* 1990)

A method of solving the axisymmetric static problem of thermoelasticity based on the use of generalized functions is proposed for a multilayered unbounded solid cylinder free of external loads, through whose surface convective heat exchange occurs with a variable heat transfer coefficient.

1. EQUATIONS WITH DISCONTINUOUS AND SINGULAR COEFFICIENTS OF THE TWO-DIMENSIONAL STATIC PROBLEM OF THERMOELASTICITY OF MULTILAYER CYLINDERS

CONSIDER a cylinder of circular transverse cross-section, free from external loads, composed of an arbitrary number of concentrically distributed layers with different physical and mechanical characteristics. The cylinder is heated by convective heat transfer from the surrounding medium of variable temperature. We will assume that the cylinders are in ideal thermomechanical contact with each other, and that the heat transfer coefficient is a function of the axial coordinate.

We will write the physical and mechanical characteristics of a multilayered cylinder as a single whole in the form [1]

$$p(r) = p_1 + \sum (p_{k+1} - p_k) S(r - r_k), \quad S(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leqslant 0 \end{cases} \tag{1.1}$$

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 1035–1040, 1991.